

# On $\Pi$ -permutable subgroups of finite groups\*

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## Abstract

Let  $\sigma = \{\sigma_i | i \in I\}$  be some partition of the set of all primes  $\mathbb{P}$  and  $\Pi$  a non-empty subset of the set  $\sigma$ . A set  $\mathcal{H}$  of subgroups of a finite group  $G$  is said to be a *complete Hall  $\Pi$ -set* of  $G$  if every member of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \Pi$  and  $\mathcal{H}$  contains exact one Hall  $\sigma_i$ -subgroup of  $G$  for every  $\sigma_i \in \Pi$  such that  $\sigma_i \cap \pi(G) \neq \emptyset$ . A subgroup  $H$  of  $G$  is called  *$\Pi$ -quasinormal* or  *$\Pi$ -permutable* in  $G$  if  $G$  possesses a complete Hall  $\Pi$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that  $AH_i^x = H_i^x A$  for any  $i$  and all  $x \in G$ . We study the embedding properties of  $H$  under the hypothesis that  $H$  is  $\Pi$ -permutable in  $G$ . Some known results are generalized.

## 1 Introduction

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. Moreover,  $\mathbb{P}$  is the set of all primes,  $\pi \subseteq \mathbb{P}$  and  $\pi' = \mathbb{P} \setminus \pi$ . If  $n$  is an integer, the symbol  $\pi(n)$  denotes the set of all primes dividing  $|n|$ ; as usual,  $\pi(G) = \pi(|G|)$ , the set of all primes dividing the order of  $G$ .

In what follows,  $\sigma = \{\sigma_i | i \in I\}$  is some partition of  $\mathbb{P}$ , that is,  $\mathbb{P} = \cup_{i \in I} \sigma_i$  and  $\sigma_i \cap \sigma_j = \emptyset$  for all  $i \neq j$ ;  $\Pi$  is always supposed to be a non-empty subset of the set  $\sigma$  and  $\Pi' = \sigma \setminus \Pi$ .

In practice, we often deal with two limited cases:  $\sigma = \{\{2\}, \{3\}, \{5\}, \dots\}$  and  $\sigma = \{\pi, \pi'\}$ .

Recall that  $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$  [1].  $G$  is called: a  *$\Pi$ -group* if  $\sigma(G) \subseteq \Pi$ ;  *$\sigma$ -primary* [2] if  $G$  is a  $\Pi$ -group for some one-element set  $\Pi$ .

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A set  $\mathcal{H}$  of subgroups of  $G$  is said to be a *complete Hall  $\Pi$ -set* of  $G$  if every member of  $\mathcal{H}$  is a Hall  $\sigma_i$ -subgroup of  $G$  for some  $\sigma_i \in \Pi$  and  $\mathcal{H}$  contains exact one Hall  $\sigma_i$ -subgroup of  $G$  for every  $\sigma_i \in \Pi \cap \sigma(G)$ . We say also that  $G$  is:  $\Pi$ -*full* if  $G$  possesses a *complete Hall  $\Pi$ -set*; a  $\Pi$ -*full group of Sylow type* if every subgroup of  $G$  is a  $D_{\sigma_i}$ -group for all  $\sigma_i \in \Pi$ .

Let  $\mathcal{L}$  be some non-empty set of subgroups of  $G$  and  $E$  a subgroup of  $G$ . Then a subgroup  $A$  of  $G$  is called  $\mathcal{L}$ -*permutable* if  $AH = HA$  for all  $H \in \mathcal{L}$ ;  $\mathcal{L}^E$ -*permutable* if  $AH^x = H^x A$  for all  $H \in \mathcal{L}$  and all  $x \in E$ .

If  $\mathcal{S}$  is a complete Sylow  $\pi$ -set of  $G$  (that is, every member of  $\mathcal{S}$  is a Sylow  $p$ -subgroup for some  $p \in \pi$  and  $\mathcal{S}$  contains exact one Sylow  $p$ -subgroup for every  $p \in \pi$ ), then an  $\mathcal{L}^G$ -permutable subgroup is called  $\pi$ -*permutable* or  $\pi$ -*quasinormal* (Kegel [3]) in  $G$ . The  $\pi(G)$ -permutable subgroups are also called  $S$ -*permutable* or  $S$ -*quasinormal*.

In this note we study the following generalization of  $\pi$ -permutability.

**Definition 1.1.** We say that a subgroup  $H$  of  $G$  is  $\Pi$ -*quasinormal* or  $\Pi$ -*permutable* in  $G$  if  $G$  possesses a complete Hall  $\Pi$ -set  $\mathcal{H}$  such that  $H$  is  $\mathcal{H}^G$ -permutable.

Before continuing, consider some examples.

**Example 1.2.** (1)  $G$  is called  $\sigma$ -*soluble* [2] if every chief factor of  $G$  is  $\sigma$ -primary. In view of Theorem A in [1], every  $\sigma$ -soluble group is a  $\Pi$ -full group of Sylow type for each  $\Pi \subseteq \sigma$ .

(2)  $G$  is called  $\sigma$ -*nilpotent* [4] if  $G$  possesses a complete Hall  $\sigma$ -set  $\mathcal{H} = \{H_1, \dots, H_t\}$  such that  $G = H_1 \times \dots \times H_t$ . Therefore every subgroup of every  $\sigma$ -nilpotent group  $G$  is  $\Pi$ -permutable in  $G$  for each  $\Pi \subseteq \sigma$ .

(3) Now let  $p > q > r$  be primes, where  $q$  divides  $p - 1$  and  $r$  divides  $q - 1$ . Let  $H = Q \rtimes R$  be a non-abelian group of order  $qr$ ,  $P$  a simple  $\mathbb{F}_p H$ -module which is faithful for  $H$ , and  $G = P \rtimes H$ . Let  $\sigma = \{\sigma_1, \sigma_2\}$ , where  $\sigma_1 = \{p, r\}$  and  $\sigma_2 = \{p, r\}'$ . Then  $G$  is not  $\sigma$ -nilpotent and  $|P| > p$ . Since  $q$  divides  $p - 1$ ,  $PQ$  is supersoluble. Hence for some normal subgroup  $L$  of  $PQ$  we have  $1 < L < P$ . Then for every Hall  $\sigma_1$ -subgroup  $V$  of  $G$  we have  $L \leq P \leq V$ , so  $LV = V = VL$ . On the other hand, for every Hall  $\sigma_2$ -subgroup  $Q^x$  of  $G$  we have  $Q^x \leq PQ$ , so  $LQ^x = Q^x L$ . Hence  $L$  is  $\sigma$ -permutable in  $G$ . It is also clear that  $L$  is not normal in  $G$ , so  $LR \neq RL$ , which implies that  $L$  is not  $S$ -permutable in  $G$ .

We will also need the following modification of the main concept in [5]: A subgroup  $A$  of  $G$  is called:  $\sigma$ -*subnormal* in  $G$  [2] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_n = G$$

such that either  $A_{i-1}$  is normal in  $A_i$  or  $A_i/(A_{i-1})_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, n$ .

In this definition  $(A_{i-1})_{A_i}$  denotes the product of all normal subgroups of  $A_i$  contained in  $A_{i-1}$ .

We use  $G^{N_\sigma}$  to denote the  $\sigma$ -*nilpotent residual* of  $G$ , that is, the intersection of all normal subgroups  $N$  of  $G$  with  $\sigma$ -nilpotent quotient  $G/N$ .

Our main goal here is to prove the following

**Theorem 1.3.** *Let  $H$  be a  $\Pi$ -subgroup of  $G$  and  $D = G^{N_\sigma}$ .*

(i) *If  $G$  is  $\Pi$ -full and possesses a complete Hall  $\Pi$ -set  $\mathcal{H}$  such that  $H$  is  $\mathcal{H}^D$ -permutable, then  $H$  is  $\sigma$ -subnormal in  $G$  and the normal closure  $H^G$  of  $H$  in  $G$  is a  $\Pi$ -group.*

(ii) *If  $H$  is  $\Pi$ -permutable in  $G$  and, in the case when  $\Pi \neq \sigma(G)$ ,  $G$  possesses a complete Hall  $\Pi'$ -set  $\mathcal{K}$  such that  $H$  is  $\mathcal{K}$ -permutable, then  $H^G/H_G$  is  $\sigma$ -nilpotent and the normalizer  $N_G(H)$  of  $H$  is also  $\Pi$ -permutable. Moreover,  $N_G(H)$  is  $\mathcal{K}^G$ -permutable for each complete Hall  $\Pi$ -set  $\mathcal{H}$  of  $G$  such that  $H$  is  $\mathcal{H}^G$ -permutable.*

(iii) *If  $G$  is a  $\Pi'$ -full group of Sylow type and  $H$  is  $\Pi'$ -permutable in  $G$ , then  $H^G$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup.*

Consider some corollaries of Theorem 1.3.

Theorem 1.3(i) immediately implies

**Corollary 1.4** (Kegel [5]). *If a  $\pi$ -subgroup  $H$  of  $G$  is  $S$ -permutable in  $G$ , then  $H$  is subnormal in  $G$ .*

Now, consider some special cases of Theorem 1.3(ii). First note that in the case when  $\sigma = \{\{2\}, \{3\}, \dots\}$  we get from Theorem 1.3(ii) the following results.

**Corollary 1.5.** *Let  $H$  be a  $\pi$ -subgroup of  $G$ . If  $H$  is  $\pi$ -permutable in  $G$  and, also,  $H$  permutes with some Sylow  $p$ -subgroup of  $G$  for each prime  $p \in \pi'$ , then the normalizer  $N_G(H)$  of  $H$  is  $\pi$ -permutable in  $G$ .*

In particular, in the case when  $\pi = \mathbb{P}$ , we have

**Corollary 1.6** (Schmid [6]). *If a subgroup  $H$  of  $G$  is  $S$ -permutable in  $G$ , then the normalizer  $N_G(H)$  of  $H$  is also  $S$ -permutable.*

**Corollary 1.7.** *Let  $H$  be a  $\pi$ -subgroup of  $G$ . If  $H$  is  $\pi$ -permutable in  $G$  and, also,  $H$  permutes with some Sylow  $p$ -subgroup of  $G$  for each prime  $p \in \pi'$ , then  $H/H_G$  is nilpotent.*

**Corollary 1.8** (Deskins [7]). *If a subgroup  $H$  of  $G$  is  $S$ -permutable in  $G$ , then  $H/H_G$  is nilpotent.*

Recall that  $G$  is said to be a  $\pi$ -decomposable if  $G = O_\pi(G) \times O_{\pi'}(G)$ , that is,  $G$  is the direct product of its Hall  $\pi$ -subgroup and Hall  $\pi'$ -subgroup.

In the case when  $\sigma = \{\pi, \pi'\}$  we get from Theorem 1.3(ii) the following

**Corollary 1.9.** *Suppose that  $G$  is  $\pi$ -separable. If a subgroup  $H$  of  $G$  permutes with all Hall  $\pi$ -subgroups of  $G$  and with Hall  $\pi'$ -subgroups of  $G$ , then  $H^G/H_G$  is  $\pi$ -decomposable.*

In particular, we have

**Corollary 1.10.** *Suppose that  $G$  is  $p$ -soluble. If a subgroup  $H$  of  $G$  permutes with all Sylow  $p$ -subgroups of  $G$  and with all  $p$ -complements of  $G$ , then  $H^G/H_G$  is  $p$ -decomposable.*

Finally, in the case when  $\Pi = \sigma$ , we get from Theorem 1.3(ii) the following

**Corollary 1.11** (Skiba [2]). *Suppose that  $G$  is a  $\sigma$ -full group and let  $H$  be a subgroup of  $G$ . If  $H$  is  $\sigma$ -permutable in  $G$ , then  $H^G/H_G$  is  $\sigma$ -nilpotent.*

From Theorem 1.3(iii) we get

**Corollary 1.12.** *Let  $H$  be a  $\pi$ -subgroup of  $G$ . If  $H$  permutes with every Sylow  $p$ -subgroup of  $G$  for  $p \in \pi'$ , then  $H^G$  possesses a nilpotent  $\pi$ -complement.*

A subgroup  $H$  of  $G$  is called a  $S$ -semipermutable in  $G$  if  $H$  permutes with all Sylow subgroups  $P$  of  $G$  such that  $(|H|, |P|) = 1$ . If  $H$  is  $S$ -semipermutable in  $G$  and  $\pi = \pi(H)$ , then  $H$  is  $\pi'$ -permutable in  $G$ . Hence from Corollary 1.12 we get the following known result.

**Corollary 1.13** (Isaacs [8]). *If a  $\pi$ -subgroup  $H$  of  $G$  is  $S$ -semipermutable in  $G$ , then  $H^G$  possesses a nilpotent  $\pi$ -complement.*

Note that in the group  $G = C_7 \rtimes \text{Aut}(C_7)$  a subgroup of order 3 is  $\pi'$ -permutable in  $G$ , where  $\pi = \{2, 3\}$ , but it is not  $S$ -semipermutable.

## 2 Preliminaries

We use:  $O^\Pi(G)$  to denote the subgroup of  $G$  generated by all its  $\Pi'$ -subgroups;  $O_\Pi(G)$  to denote the subgroup of  $G$  generated by all its normal  $\Pi$ -subgroups. A subgroup  $H$  of  $G$  is said to be: a *Hall  $\Pi$ -subgroup* of  $G$  [1] if  $|H|$  is a  $\Pi$ -number (that is,  $\pi(H) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$ ) and  $|G : H|$  is a  $\Pi'$ -number.

**Lemma 2.1.** *Let  $A$ ,  $K$  and  $N$  be subgroups of  $G$ . Suppose that  $A$  is  $\sigma$ -subnormal in  $G$  and  $N$  is normal in  $G$ .*

- (1)  $A \cap K$  is  $\sigma$ -subnormal in  $K$ .
- (2) If  $K$  is a  $\sigma$ -subnormal subgroup of  $A$ , then  $K$  is  $\sigma$ -subnormal in  $G$ .
- (3) If  $K$  is  $\sigma$ -subnormal in  $G$ , then  $A \cap K$  and  $\langle A, K \rangle$  are  $\sigma$ -subnormal in  $G$ .
- (4)  $AN/N$  is  $\sigma$ -subnormal in  $G/N$ .
- (5) If  $N \leq K$  and  $K/N$  is  $\sigma$ -subnormal in  $G/N$ , then  $K$  is  $\sigma$ -subnormal in  $G$ .
- (6) If  $K \leq A$  and  $A$  is  $\sigma$ -nilpotent, then  $K$  is  $\sigma$ -subnormal in  $G$ .
- (7) If  $H \neq 1$  is a Hall  $\Pi$ -subgroup of  $G$  and  $A$  is not a  $\Pi'$ -group, then  $A \cap H \neq 1$  is a Hall  $\Pi$ -subgroup of  $A$ .
- (8) If  $|G : A|$  is a  $\Pi$ -number, then  $O^\Pi(A) = O^\Pi(G)$ .
- (9) If  $G$  is  $\Pi$ -full and  $A$  is a  $\Pi$ -group, then  $A \leq O_\Pi(G)$ .

**Proof.** Statements (1)–(8) are known [2, Lemma 2.6]).

(9) Assume that this assertion is false and let  $G$  be a counterexample of minimal order. By hypothesis, there is a subgroup chain  $A = A_0 \leq A_1 \leq \dots \leq A_r = G$  such that either  $A_{i-1}$  is normal in  $A_i$  or  $A_i/(A_{i-1})_{A_i}$  is  $\sigma$ -primary for all  $i = 1, \dots, r$ . Let  $M = A_{r-1}$ . We can assume without loss

of generality that  $M \neq G$ . Let  $D = A \cap M_G$ .

First note that  $A$  is not  $\sigma$ -primary. Indeed, assume that  $A$  is a  $\sigma_i$ -group. By hypothesis,  $G$  has a Hall  $\sigma_i$ -subgroup, say  $H$ . Then, by Assertion (7), for any  $x \in G$  we have  $A \leq H^x$ . Hence  $A^G \leq H_G \leq O_\Pi(G)$ , a contradiction. Hence  $|\sigma(A)| > 1$ .

Suppose that  $D \neq 1$ . The subgroup  $D$  is  $\sigma$ -subnormal in  $M_G$  by Lemma 2.1(1)(3), so the choice of  $G$  implies that  $D \leq O_\Pi(M_G)$ . Hence  $O_\Pi(M_G) \neq 1$ . But since  $O_\Pi(M_G)$  is characteristic in  $M_G$ , we have that  $O_\Pi(M_G) \leq O_\Pi(G)$ . The hypothesis holds for  $(G/O_\Pi(G), AO_\Pi(G)/O_\Pi(G))$  by Assertion (4). Therefore  $AO_\Pi(G)/O_\Pi(G) \leq O_\Pi(G/O_\Pi(G)) = 1$ . It follows that  $A \leq O_\Pi(G)$ , a contradiction. Hence  $A \cap M_G = 1$ , so  $M$  is not normal in  $G$ . Therefore,  $G/M_G$  is a  $\sigma_j$ -group for some  $j \in I$ . But then  $A \simeq AM_G/M_G$  is  $\sigma$ -primary. This contradiction completes the proof.

The first three statements in the next lemma can be proved by the direct calculations and the last statement see [9, A, 1.6(a)].

**Lemma 2.2.** *Let  $H$ ,  $K$  and  $N$  be subgroups of  $G$ . Let  $\mathcal{H} = \{H_1, \dots, H_t\}$  be a complete Hall  $\Pi$ -set of  $G$  and  $\mathcal{L} = \mathcal{H}^K$ . Suppose that  $H$  is  $\mathcal{L}$ -permutable and  $N$  is normal in  $G$ .*

(1) *If  $H \leq E \leq G$ , then  $H$  is  $\mathcal{L}^*$ -permutable, where  $\mathcal{L}^* = \{H_1 \cap E, \dots, H_t \cap E\}^{K \cap E}$ . In particular, if  $H$  is  $\Pi$ -permutable in  $G$  and either  $G$  is a  $\Pi$ -full group of Sylow type or  $E$  is normal in  $G$ , then  $H$  is  $\Pi$ -permutable in  $E$ .*

(2) *The subgroup  $HN/N$  is  $\mathcal{L}^{**}$ -permutable, where  $\mathcal{L}^{**} = \{H_1 N/N, \dots, H_t N/N\}^{KN/N}$ .*

(3) *If  $G$  is a  $\Pi$ -full group of Sylow type and  $E/N$  is a  $\Pi$ -permutable subgroup of  $G/N$ , then  $E$  is  $\Pi$ -permutable in  $G$ .*

(4) *If  $K$  is  $\mathcal{L}$ -permutable, then  $\langle H, K \rangle$  is  $\mathcal{L}$ -permutable.*

**Lemma 2.3** (See Lemma 2.2 in [1]). *Let  $H$  be a normal subgroup of  $G$ . If  $H/H \cap \Phi(G)$  is a  $\Pi$ -group, then  $H$  has a Hall  $\Pi$ -subgroup, say  $E$ , and  $E$  is normal in  $G$ .*

We say that a group  $G$  is  $\Pi$ -closed if  $O_\Pi(G)$  is a Hall  $\Pi$ -subgroup of  $G$ . Two integers  $n$  and  $m$  are called  $\sigma$ -coprime if  $\sigma(n) \cap \sigma(m) = \emptyset$ .

**Lemma 2.4.** *If a  $\sigma$ -soluble group  $G$  has three  $\Pi$ -closed subgroups  $A$ ,  $B$  and  $C$  whose indices  $|G : A|$ ,  $|G : B|$ ,  $|G : C|$  are pairwise  $\sigma$ -coprime, then  $G$  is  $\Pi$ -closed.*

**Proof.** Suppose that this lemma is false and let  $G$  be a counterexample with  $|G|$  minimal. Let  $N$  be a minimal normal subgroup of  $G$ . Then the hypothesis holds for  $G/N$ , so  $G/N$  is  $\Pi$ -closed by the choice of  $G$ . Therefore  $N$  is not a  $\Pi$ -group. Moreover,  $N$  is the unique minimal normal subgroup of  $G$  and, by Lemma 2.3,  $N \not\leq \Phi(G)$ . Hence  $C_G(N) \leq N$ . Since  $G$  is  $\sigma$ -soluble by hypothesis,  $N$  is  $\sigma$ -primary, say  $N$  is a  $\sigma_i$ -group. Then  $\sigma_i \in \Pi'$ .

Since  $|G : A|$ ,  $|G : B|$ ,  $|G : C|$  are pairwise  $\sigma$ -coprime, there are at least two subgroups, say  $A$  and  $B$ , such that  $N \leq A \cap B$ . Then  $O_\Pi(A) \leq C_G(N) \leq N$ , so  $O_\Pi(A) = 1$ . But by hypothesis,  $A$  is  $\Pi$ -closed, hence  $A$  is a  $\Pi'$ -group. Similarly we get that  $B$  is a  $\Pi'$ -group and so  $G = AB$  is a  $\Pi'$ -group.

But then  $G$  is  $\Pi$ -closed. This contradiction completes the proof of the lemma.

Recall that  $G$  is called a *Schmidt group* if  $G$  is not nilpotent but every proper subgroup of  $G$  is nilpotent.

**Proposition 2.5.** *Let  $G$  be a  $\sigma$ -soluble group. Suppose that  $G$  is not  $\sigma'_i$ -closed but all proper subgroups of  $G$  are  $\sigma'_i$ -closed. Then  $G$  is a  $\sigma_i$ -closed Schmidt group.*

**Proof.** Suppose that this proposition is false and let  $G$  be a counterexample of minimal order. Let  $R$  be a minimal normal subgroup of  $G$  and  $\{H_1, \dots, H_t\}$  a complete Hall  $\sigma$ -set of  $G$ . Without loss of generality we can assume that  $H_1$  is a  $\sigma_i$ -group.

(1)  $|\sigma(G)| = 2$ . Hence  $G = H_1 H_2$ .

It is clear that  $|\sigma(G)| > 1$ . Suppose that  $|\sigma(G)| > 2$ . Then, since  $G$  is  $\sigma$ -soluble, there are maximal subgroups  $M_1$ ,  $M_2$  and  $M_3$  whose indices  $|G : M_1|$ ,  $|G : M_2|$  and  $|G : M_3|$  are  $\sigma$ -coprime. Hence  $G = M_1 M_2 = M_2 M_3 = M_1 M_3$ . But the subgroups  $M_1$ ,  $M_2$  and  $M_3$  are  $\sigma'_i$ -closed by hypothesis. Hence  $G$  is  $\sigma'_i$ -closed by Lemma 2.4, a contradiction. Thus  $|\sigma(G)| = 2$ .

(2) If either  $R \leq \Phi(G)$  or  $R \leq H_2$ , then  $G/R$  is a  $\sigma_i$ -closed Schmidt group.

Lemma 2.3 and the choice of  $G$  imply that  $G/R$  is not  $\sigma'_i$ -closed. On the other hand, every maximal subgroup  $M/R$  of  $G/R$  is  $\sigma'_i$ -closed since  $M$  is  $\sigma'_i$ -closed. Hence the hypothesis holds for  $G/R$ . The choice of  $G$  implies that  $G/R$  is a  $\sigma_i$ -closed Schmidt group.

(3)  $\Phi(G) = 1$ ,  $R$  is the unique minimal normal subgroup of  $G$  and  $R \leq H_1$ .

Suppose that  $R \leq \Phi(G)$ . Then  $R$  is a  $r$ -group for some prime  $r$  and, in view of Claim (1), Lemma 2.3 and [10, IV, 5.4],  $G = H_1 \rtimes H_2 = P \rtimes Q$ , where  $H_1 = P$  is a  $p$ -group and  $H_2 = Q$  is a  $q$ -group for some different primes  $p$  and  $q$ . Assume that  $R \leq Q$  and take a subgroup  $L$  of order  $q$  in  $R \cap Z(Q)$ . Then it is clear that  $R < Q$ , so  $PR < G$  and hence  $PR = P \times Q$  is  $p$ -nilpotent. Therefore  $L \leq Z(G)$ , so  $R = L \leq Z(G)$ . But for every maximal subgroup  $M$  of  $G$  we have  $R \leq M$  and  $M/R$  is nilpotent. Hence every maximal subgroup of  $G$  is nilpotent and so  $G$  is a  $\sigma_i$ -closed Schmidt group, a contradiction. Similarly, we get that  $G$  is a  $\sigma_i$ -closed Schmidt group in the case when  $R \leq P$ . Therefore  $R \not\leq \Phi(G)$ .

Now assume that  $G$  has a minimal normal subgroup  $L \neq R$ . Then by (3), there are maximal subgroups  $M$  and  $T$  of  $G$  such that  $LM = G$  and  $RT = G$ . By hypothesis,  $M$  and  $T$  are  $\sigma'_i$ -closed. Hence  $G/L \simeq LM/L \simeq M/M \cap L$  is  $\sigma'_i$ -closed. Similarly,  $G/R$  is  $\sigma'_i$ -closed and so  $G \simeq G/L \cap R$  is  $\sigma_i$ -nilpotent, a contradiction. Hence  $R$  is the unique minimal normal subgroup of  $G$ , and so  $R \leq H_1$ .

*Final contradiction.* In view of Claim (3),  $C_G(R) \leq R$ . Hence  $|H_2|$  is a prime and  $RH_2 = G$  since  $R \leq H_1$  and every proper subgroup of  $G$  is  $\sigma'_i$ -closed. Therefore  $R = H_1$ , so  $R$  is not abelian since  $G$  is not a  $\sigma_i$ -closed Schmidt group. By Claim (1) and Theorem 3.5 in [11], for any prime  $p$  dividing  $|R|$  there is a Sylow  $p$ -subgroup  $P$  of  $G$  such that  $PH_2 = H_2P$ . But  $H_2P < G$ , so  $H_2P = H_2 \rtimes P$ . This implies that  $R \leq N_G(H_2)$  and thereby  $G = R \times H_2 = H_1 \times H_2$ . This final contradiction completes the proof of the result.

**Corollary 2.6.** *Let  $G$  be a minimal non- $\sigma$ -nilpotent group, that is,  $G$  is not  $\sigma$ -nilpotent, but every proper subgroup of  $G$  is  $\sigma$ -nilpotent. If  $G$  is a  $\sigma$ -soluble, then  $G$  is a Schmidt group.*

**Proof.** It is clear that  $G$  is  $\sigma$ -nilpotent if and only if  $G$  is  $\sigma'_i$ -closed for all  $\sigma_i \in \sigma$ . Hence, for some  $i$ ,  $G$  is not  $\sigma'_i$ -closed. On the other hand, every proper subgroup of  $G$  is  $\sigma'_i$ -closed. Hence  $G$  is a Schmidt group by Proposition 2.5.

**Proposition 2.7.** *Let  $G$  be a  $\Pi$ -full group of Sylow type. If  $G$  possesses a  $\sigma$ -nilpotent Hall  $\Pi$ -subgroup  $H$ , then every  $\sigma$ -soluble  $\Pi$ -subgroup of  $G$  is contained in a conjugate of  $H$ . In particular, any two  $\sigma$ -soluble Hall  $\Pi$ -subgroups of  $G$  are conjugate.*

**Proof.** Suppose that this proposition is false and let  $G$  be a counterexample of minimal order. Then some  $\sigma$ -soluble  $\Pi$ -subgroup  $K$  of  $G$  is not contained in  $H^x$  for all  $x \in G$ . We can assume without loss of generality that every proper subgroup  $V$  of  $K$  is contained in a conjugate of  $H$ , so  $V$  is  $\sigma$ -nilpotent. Hence either  $K$  is  $\sigma$ -nilpotent or  $K$  is a minimal non- $\sigma$ -nilpotent group. Then in view of Corollary 2.6 and [10, IV, 5.4],  $K$  has a normal Hall  $\sigma_i$ -subgroup  $L$  for some  $\sigma_i \in \sigma(K)$ . Now arguing as in the proof of Wielandt's theorem [12, (10.1.9)], one can show that for some  $y \in G$  we have  $K \leq H^y$ . This contradiction completes the proof of the result.

**Corollary 2.8.** *Let  $G$  be a  $\Pi$ -full group of Sylow type. Suppose that every chief factor of  $G$  possesses a  $\sigma$ -nilpotent Hall  $\Pi$ -subgroup. Then  $G$  possesses a  $\sigma$ -soluble Hall  $\Pi$ -subgroup.*

**Proof.** Let  $R$  be a minimal normal subgroup of  $G$ ,  $H$  a  $\sigma$ -nilpotent Hall  $\Pi$ -subgroup of  $R$  and  $N = N_G(H)$ . By induction,  $G/R$  has a  $\sigma$ -soluble Hall  $\Pi$ -subgroup, say  $U/R$ . Therefore if  $R$  is a  $\Pi$ -group, then  $U$  is a  $\sigma$ -soluble Hall  $\Pi$ -subgroup of  $G$ . On the other hand, if  $R$  is a  $\Pi'$ -group, then  $U = R \rtimes V$  by the Schur-Zassenhaus theorem, where  $V \simeq U/R$  is a  $\sigma$ -soluble Hall  $\Pi$ -subgroup of  $G$ . Now suppose that  $1 < H < R$ . Proposition 2.7 and the Fattini argument imply that  $G = RN$ , where  $|G : N| = |R/R \cap N|$  is a  $\Pi'$ -number and  $N < G$ . Then  $N/N \cap R \simeq G/R$  possesses a  $\sigma$ -soluble Hall  $\Pi$ -subgroup. Hence in view Proposition 2.7, the hypothesis holds for  $N$ , so  $N$  possesses a  $\sigma$ -soluble Hall  $\Pi$ -subgroup  $W$  by induction. It is clear now that  $W$  is a Hall  $\Pi$ -subgroup of  $G$ . The corollary is proved.

### 3 Proof of Theorem 1.3

Suppose that this theorem is false and let  $(G, H)$  be a counterexample with  $|G| + |G : H|$  as small as possible. Then  $H \neq H^G$ .

(i), (ii) By hypothesis,  $G$  possesses a complete Hall  $\Pi$ -set, say  $\mathcal{H} = \{H_1, \dots, H_t\}$ . We can assume without loss of generality that  $H_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ . Let  $E = H_1^G \cdots H_t^G$ .

Suppose that Assertion (i) is false. Then in view of Lemma 2.1(9),  $H$  is not  $\sigma$ -subnormal in  $G$ . Moreover, in this case we have  $E = G$ . Indeed, since the class of all  $\sigma$ -nilpotent groups is closed under taking subgroups, homomorphic images and the direct products,  $E/E \cap D \simeq DE/D$  is  $\sigma$ -nilpotent.

Hence  $E^{N\sigma} \leq D$ . It follows that the hypothesis holds for  $(E, H)$ . Thus in the case when  $E < G$  the choice of  $(G, H)$  implies that  $H$  is  $\sigma$ -subnormal in  $E$  and so  $H$  is  $\sigma$ -subnormal in  $G$ , a contradiction. Therefore  $E = G$ . Since  $H \neq H^G$ , it follows that for some  $x \in G$  and  $H_i \in \mathcal{H}$  we have  $H_i^x \not\leq N_G(H)$ . Now, arguing as in Claim (2) of the proof of Theorem B in [2], one can show that  $H$  is  $\sigma$ -subnormal in  $G$ . This contradiction completes the proof of (i).

(ii) Suppose that this assertion is false. Then:

(1) *The hypothesis holds for  $(G/H_G, H/H_G)$ , so  $H_G = 1$ .*

First note that the hypothesis holds for  $(G/H_G, H/H_G)$  by Lemma 2.2(2). Assume that  $H_G \neq 1$ . Then the choice of  $(G, H)$  implies that  $H^G/H_G$  is  $\sigma$ -nilpotent and  $N_{G/H_G}(H/H_G) = N_G(H)/H_G$  is  $\mathcal{H}^*$ -permutable by Lemma 2.2(2), where

$$\mathcal{H}^* = \{H_1 H_G/H_G, \dots, H_t H_G/H_G\}^{G/H_G}.$$

But then, clearly,  $N_G(H)$  is  $\mathcal{H}^G$ -permutable. This shows that Assertion (ii) is true. Therefore the choice of  $(G, H)$  implies that  $H_G = 1$ .

(2)  $t > 1$ .

Assume that  $t = 1$ , that is,  $H$  is a  $\sigma_1$ -group. Then  $HH_1^x = H_1^x H = H_1^x$  for all  $x \in G$ , so  $H^G \leq (H_1)_G \leq O_{\sigma_1}(G)$ , which implies that  $H^G$  is  $\sigma$ -nilpotent. Hence  $H$  is  $\sigma$ -subnormal in  $G$  by Lemma 2.1(6). Note also that for any Hall  $\sigma_1'$ -subgroup  $V$  of  $G$  such that  $HV = VH$  we have  $H = VH \cap O_{\sigma_1}(G)$ , so  $V \leq N_G(H)$ . Therefore if  $H$  is  $\Pi$ -permutable in  $G$  and also, in the case when  $\Pi \neq \sigma(G)$ ,  $H$  is  $\mathcal{K}$ -permutable, then  $|G : N_G(H)|$  is a  $\sigma_1$ -number, which implies that  $N_G(H)H_1^x = G = H_1^x N_G(H)$  for all  $x \in G$ . This means that  $N_G(H)$  is  $\Pi$ -permutable in  $G$ . Thus Assertion (ii) is true, a contradiction. Therefore  $t > 1$ .

Let  $L_i = O^{\sigma_i'}(H)$ , for all  $i = 1, \dots, t$ . Then  $H = L_1 \cdots L_t$  and  $N_G(H) = N_G(L_1) \cap \cdots \cap N_G(L_t)$ . Let

$$W_i = H_1^G \cdots H_{i-1}^G H_{i+1}^G \cdots H_t^G,$$

for all  $i = 1, \dots, t$ , and  $W = W_1 \cap \cdots \cap W_t$ .

(3)  $W_i \leq N_G(L_i)$  for all  $i = 1, \dots, t$ , so  $W \leq N_G(H)$ .

Indeed, since  $H$  is  $\sigma$ -subnormal in  $G$  by Part (i), Lemma 2.1(8) implies that  $H_i^x \leq N_G(O^{\sigma_i}(H))$  for all  $x \in G$ . This means that  $H_i^G \leq N_G(O^{\sigma_i}(H))$ . Hence  $H_i^G \leq N_G(L_j)$  for all  $j \neq i$ , so  $W_i \leq N_G(L_i)$  for all  $i = 1, \dots, t$ .

(4)  $H^G$  is  $\sigma$ -nilpotent.

Suppose that this is false. Let  $K = H_1 \cdots H_t W$ . Then:

(a)  $K$  is a subgroup of  $G$ ,  $H \leq K$  and  $|K : W|$  is a  $\Pi$ -number.

First note that  $(H_i W/W)^{G/W} = H_i^G W/W$  and



$$\begin{aligned}
WW_i \cap H_i^G W &= W(W_i \cap H_i^G W) = W(W_i \cap H_i^G(W_1 \cap \cdots \cap W_t)) = \\
&= W(W_i \cap W_1 \cap \cdots \cap W_{i-1} \cap W_{i+1} \cap \cdots \cap W_t \cap W_i H_i^G) = W(W \cap E) = W.
\end{aligned}$$

Therefore

$$E/W = (H_1 W/W)^{G/W} \times \cdots \times (H_t W/W)^{G/W}.$$

This means that  $[H_i W/W, H_j W/W] = 1$ , for all  $i \neq j$ . Hence  $K = H_1 \cdots H_t W = (H_1 W) \cdots (H_t W)$  is the product of pairwise permutable subgroups, which implies that  $K$  is a subgroup of  $G$ . It is also clear that  $K/W$  is a Hall  $\Pi$ -subgroup of  $G/W$ . Hence  $|K : W|$  is a  $\Pi$ -number and  $WH/W \leq K/W$  by Lemma 2.1(4)(7), so we have (a).

(b) *The hypothesis holds for  $(K, H)$ .*

Let  $\mathcal{K} = \{K_1, \dots, K_n\}$ . Since  $|K : W|$  is a  $\Pi$ -number,  $K_i \cap K$  is a Hall  $\sigma_i$ -subgroup of  $K$  and hence  $\mathcal{B} = \{K_1 \cap K, \dots, K_n \cap K\}$  is a complete Hall  $\Pi'$ -set of  $K$ . On the other hand, for any  $K_i \in \mathcal{K}$  we have  $HK_i \cap K = (K_i \cap K)H$  and so  $H$  is  $\mathcal{B}$ -permutable. Finally, it is clear that  $H$  is  $\Pi$ -permutable in  $K$ . Hence the hypothesis holds for  $(K, H)$ .

(c)  $K < G$ .

Suppose that  $K = G$ . Then, since  $|K : W| = |G : W|$  is a  $\Pi$ -number by Claim (4), for every  $K_i \in \mathcal{K}$  and every  $x \in G$  we have  $K_i^x \leq W \leq N_G(H)$  by Claim (3), so  $K_i^x H = HK_i^x$ . Therefore  $H$  is  $\sigma$ -permutable in  $G$  and so  $H^G \simeq H^G/H_G$  is  $\sigma$ -nilpotent by Theorem B in [2], contrary to our assumption on  $H$ . Hence  $K < G$ .

(d)  $|G : N_G(H)|$  is a  $\Pi$ -number (Since  $H$  is a  $\sigma$ -subnormal  $\Pi$ -subgroup of  $G$ , this follows from Lemma 2.1(8)).

(e) Conclusion for (4).

Since  $K < G$  by Claim (c), we have that  $H^K/H_K$  is  $\sigma$ -nilpotent. Because  $|G : N_G(H)|$  is a  $\Pi$ -number by Claim (d),  $G = KN_G(H)$ . Hence  $H^G \simeq H/1 = H^G/H_G = H^K/H_K$  is  $\sigma$ -nilpotent. This contradiction shows that  $H^G$  is  $\sigma$ -nilpotent.

*Final contradiction for (ii).*

Since  $H^G$  is  $\sigma$ -nilpotent by (4),  $H$  is also  $\sigma$ -nilpotent. Hence  $H$  possesses a complete Hall  $\sigma$ -set  $\{V_1, \dots, V_t\}$  such that  $H = V_1 \times \cdots \times V_t$ . Without loss of generality we can assume that  $V_i$  is a  $\sigma_i$ -group for all  $i = 1, \dots, t$ . Let  $N = N_G(H)$  and  $N_i = N_G(V_i)$ . Then  $N = N_1 \cap \cdots \cap N_t$ . Moreover, it is clear that  $L_i = V_i$  for all  $i = 1, \dots, t$ . Hence  $W_i \leq N_G(V_i)$  for all  $i = 1, \dots, t$  by Claim (3). It is also clear that  $|G : N_i|$  is a  $\sigma_i$ -number, so  $G = N_i H_i$ . Hence for any  $x \in G$  and  $H_i \in \mathcal{H}$  we have

$$\begin{aligned}
NH_i^x &= (N_1 \cap \cdots \cap N_t)H_i^x = N_i H_i^x \cap N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t = \\
&= G \cap N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t = N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t = H_i^x N
\end{aligned}$$

and so  $N$  is  $\mathcal{H}^G$ -permutable. Therefore Assertion (ii) is true. This contradiction completes the proof of Assertion (ii).

(iii) Let  $\mathcal{L} = \{L_1, \dots, L_m\}$  be a complete Hall  $\Pi'$ -set of  $G$  such that  $H$  is  $\mathcal{L}^G$ -permutable. Let  $E = H^G$  and  $R$  a minimal normal subgroup of  $G$ . First note that  $m > 1$ . Indeed, if  $m = 1$ , then  $L_1 \cap E$  is a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup of  $G$ , which contradicts the choice of  $(G, H)$ .

(1)  $ER/R$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup  $U/R$ . Therefore  $R \leq E$ .

From Lemma 2.2(2) and the choice of  $G$  it follows that  $(HR/R)^{G/R} = ER/R$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup, say  $U/R$ . Therefore, if  $R \not\leq E$ , then  $E \simeq ER/R$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup, a contradiction. Hence we have (1).

(2)  $O_\Pi(G) = 1$ .

Assume that  $R \leq O_\Pi(G)$ . Then, by the Schur-Zassenhaus theorem,  $R$  has a complement  $V$  in  $U$ , so  $V \simeq U/R$  is a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup of  $E$ , a contradiction. Hence we have (2).

(3)  $L_i^G \not\leq C_G(E)$  for all  $i = 1, \dots, t$ .

Assume that  $L_i^G \leq C_G(E)$  and let  $N$  be a minimal normal subgroup of  $G$  contained in  $L_i^G$ . Then  $N \leq E$  and  $E/N$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup, say  $U/N$ , by Claim (1). On the other hand,  $N \leq Z(U)$ , so  $U$  is  $\sigma$ -nilpotent. But a Hall  $\Pi'$ -subgroup of  $U$  is a Hall  $\Pi'$ -subgroup of  $E$ , a contradiction. Hence we have (3).

(4)  $R$  is the unique minimal normal subgroup of  $G$ .

Suppose that  $G$  has a minimal normal subgroup  $N \neq R$ . Then  $N \leq E$  and  $G/N$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup by Claim (1). Therefore  $(E/R) \times (E/N)$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup  $V$ . But  $E \simeq K \leq (E/R) \times (E/N)$  since  $R \cap N = 1$ . Hence  $E$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup. Moreover, since  $N \simeq RN/R$  possesses a  $\sigma$ -nilpotent Hall  $\Pi'$ -subgroup,  $E$  possesses a Hall  $\Pi'$ -subgroup  $U$  by Corollary 2.8. But then, by Proposition 2.7, for some  $x \in G$  we have  $U \leq V^x$  and so  $U$  is  $\sigma$ -nilpotent, contrary to the choice of  $G$ . Hence we have (4).

*Final contradiction for (iii).*

Let  $x, y \in G$  and  $A = H^x$ . Then

$$AL_i^y = (HL_i^{yx^{-1}})^x = (L_i^{yx^{-1}}H)^x = L_i^y A$$

by hypothesis. Let  $L = A^{L_i} \cap L_i^A$ . Then  $L$  is a subnormal subgroup of  $G$  by [13, 7.2.5]. Suppose that  $L \neq 1$  and let  $L_0$  be a minimal subnormal subgroup of  $G$  contained in  $L$ . Then  $V = L_0 \cap L_i$  is a Hall  $\Pi'$ -subgroup of  $L_0$  since  $L \leq AL_i$ . Moreover, in view of Claim (2),  $V \neq 1$  (see, for example, [14, Chapter 1, Lemma 5.35(5)]). We now show that  $L_i \cap R$  is a non-identity Hall  $\Pi'$ -subgroup of  $R$ . Indeed, if  $L_0$  is abelian, then  $L_0 \leq O_{\sigma_i}(G)$ , where  $\sigma_i = \pi(L_i)$ , so  $R$  is a  $\sigma_i$ -group by Claim (4). On the other hand, if  $L_0$  is non-abelian,  $L_0^G$  is a minimal normal subgroup of  $G$  and so, by Claim (4),  $L_i \cap R$  is a non-identity Hall  $\Pi'$ -subgroup of  $R$ .

Since  $m > 1$ , Claim (2) implies that there is  $j \neq i$  such that for every  $x, y \in G$  we have

$(L_j^y)^{H^x} \cap (H^x)^{L_j^y} = 1$  and so

$$[L_j^y, H^x] \leq [(L_j^y)^{H^x}, (H^x)^{L_j^y}] = 1.$$

Therefore  $L_j^G \leq C_G(E)$ , contrary Claim (3). Hence Statement (iii) holds.

The theorem is proved.

## References

- [1] A. N. Skiba, A generalization of a Hall theorem, *J. Algebra and its Application*, DOI: 10.1142/S0219498816500857.
- [2] A. N. Skiba, On  $\sigma$ -subnormal and  $\sigma$ -permutable subgroups of finite groups, *J. Algebra*, **436** (2015), 1-16.
- [3] O. H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.*, **78** (1962), 205–221.
- [4] W. Guo, A.N. Skiba, Finite groups with permutable complete Wielandt sets of subgroups, *J. Group Theory*, **18** (2015), 191-200
- [5] O. H. Kegel, Untergruppenverbände endlicher Gruppen, die den subnormalteilerverband each enthalten, *Arch. Math.*, **30**(3) (1978), 225–228.
- [6] P. Schmid, Subgroups permutable with all Sylow subgroups, *J. Algebra*, **207** (1998), 285–293.
- [7] W. E. Deskins, On quasinormal subgroups of finite groups, *Math. Z.*, **82** (1963), 125–132.
- [8] I. M. Isaacs, Semipermutable  $\pi$ -subgroups, *Arch. Math.* **102** (2014), 1–6.
- [9] K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.
- [10] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
- [11] D. Gorenstein, *Finite Groups*, Harper & Row Publishers, New York-Evanston-London, 1968.
- [12] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, Berlin-New York, 1982.
- [13] J. C. Lennox, S. E. Stonehewer, *Subnormal Subgroups of Groups*, Clarendon Press, Oxford, 1987.
- [14] W. Guo, *Structure Theory for Canonical Classes of Finite Groups*, Springer, 2015.